Assignment 4

1. (a)

$$X(u,v) = (a\sinh v\cos u,\ a\sinh v\sin u,\ au)$$
$$X_u = (-a\sinh v\sin u,\ a\sinh v\cos u,\ a)$$
$$X_v = (a\cosh v\cos u,\ a\cosh v\sin u,\ 0)$$
$$\vec{N} = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(-\sin u,\ \cos u,\ -\sinh v)}{\sqrt{1+\sinh^2 v}} = \frac{(-\sin u, \cos u, -\sinh v)}{\cosh v}$$
$$X_{uu} = (-a\sinh v\cos u,\ -a\sinh v\sin u,\ 0)$$
$$X_{uv} = (-a\cosh v\sin u,\ a\cosh v\cdot\cos u,\ 0)$$
$$X_{vv} = (a\sinh v\cdot\cos u,\ a\sinh v\cdot\sin u, 0)$$
$$g_{ij} = \begin{bmatrix} a^2\cosh^2 v & 0 \\ 0 & a^2\cosh^2 v \end{bmatrix}$$
$$h_{ij} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

$\therefore$ the mean curvature is

$$H = \frac{1}{2}\sum_{i,j} g^{ij} h_{ij} = 0$$

(b)

$$X_u = (1-u^2+v^2,\ 2uv,\ 2u)$$
$$X_v = (2uv,\ 1-v^2+u^2,\ -2v)$$
$$\vec{N} = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(-2u,\ 2v,\ 1-u^2-v^2)}{1+u^2+v^2}$$
$$X_{uu} = (-2u,\ 2v,\ 2)$$
$$X_{uv} = (2v,\ 2u,\ 0)$$
$$X_{vv} = (2u,\ -2v,\ -2)$$
$$g_{ij} = \begin{bmatrix} (1+u^2+v^2)^2 & 0 \\ 0 & (1+u^2+v^2)^2 \end{bmatrix}$$
$$h_{ij} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\therefore$ the mean curvature is

$$H = \frac{1}{2}\sum_{i,j} g^{ij} h_{ij} = 0$$

2. $X(u,v) = (u,\ h(u)\cos v,\ h(u)\sin v)$

$X_u = (1,\ h'\cos v,\ h'\sin v)$

$X_v = (0,\ -h\sin v,\ h\cos v)$

$$\vec{N} = \frac{(h',\ -\cos v,\ -\sin v)}{\sqrt{1+(h')^2}}$$

$X_{uu} = (0,\ h''\cos v,\ h''\sin v)$

$X_{uv} = (0,\ -h'\sin v,\ h'\cos v)$

$X_{vv} = (0,\ -h\cos v,\ -h\sin v)$

$$\therefore\ g_{ij} = \begin{bmatrix} 1+(h')^2 & 0 \\ 0 & h^2 \end{bmatrix}$$

$$h_{ij} = \begin{bmatrix} -h'' & 0 \\ 0 & h \end{bmatrix} \cdot \frac{1}{\sqrt{1+(h')^2}}$$

$\therefore$ the mean curvature is

$$H = \frac{1}{2}\sum_{i,j} g^{ij} h_{ij} = \frac{1}{2}\left[\frac{-h''}{1+(h')^2} + \frac{h}{h^2}\right]\cdot\frac{1}{\sqrt{1+(h')^2}}$$

$$= \frac{1}{2}\left[-\frac{h''}{[1+(h')^2]^{\frac{3}{2}}} + \frac{1}{h[1+(h')^2]^{\frac{1}{2}}}\right]$$

Next we compute for some constant a,

$$\frac{d}{du}\left[h^2 + \frac{2ah}{\sqrt{1+(h')^2}}\right]$$

$$= 2hh' + \frac{2ah'}{\sqrt{1+(h')^2}} - \frac{2ah\cdot 2h'h''}{2[1+(h')^2]^{\frac{3}{2}}}$$

$$= 2hh' + 2ahh'\left[\frac{1}{h\sqrt{1+(h')^2}} - \frac{h''}{[1+(h')^2]^{\frac{3}{2}}}\right]$$

$$= 2hh' + 2ahh'\cdot 2H$$

$$= 2hh'[1+2aH]$$

$\therefore$ H is a non-zero constant

$\Leftrightarrow$ $H = -\frac{1}{2a}$ for some constant a

$\Leftrightarrow$ $1+2aH=0$ for some constant a

"$\Rightarrow$" $\therefore$ if H is a non-zero constant

then $\frac{d}{du}\left[h^2 + \frac{2ah}{\sqrt{1+(h')^2}}\right] = 0$

$\therefore$ $h^2 + \frac{2ah}{\cdots} = b$ for some constant b

"$\Leftarrow$" if $\quad h^2 + \frac{2ah}{\sqrt{1+(h')^2}} = b$ for some constants a, b where $a \neq 0$

then $\frac{d}{du}\left[h^2 + \frac{2ah}{\sqrt{1+(h')^2}}\right] = 2hh'[1+2aH] = 0$

$\therefore hh'[1+2aH] = 0$

$\because h \neq 0$, otherwise M is not a regular surface

$\therefore \quad h'[1+2aH] = 0$

if $1+2aH(x_0) \neq 0$ for some x_0

then $1+2aH(x) \neq 0$ for x near to x_0

$\therefore \quad h'(x) = 0$ for x near to x_0

$\therefore \quad H(x) = \frac{1}{2}\left[-\frac{h''}{[1+(h')^2]^{\frac{3}{2}}} + \frac{1}{h\sqrt{1+(h')^2}}\right] = \frac{1}{2h(x)} = \frac{1}{2h(x_0)}$ for x near to x_0.

$\therefore$ H is a non-zero constant on $\{x \mid 1+2aH(x) = 0\}$

H is also a non-zero constant on each connected component of $\{x \mid 1+2aH(x) \neq 0\}$

$\because$ H is a continuous function

$\therefore$ H is globally a non-zero constant.

3. (a) $\because F=0$

$\therefore$

$$A=\begin{vmatrix} -\frac{1}{2}E_{vv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & -\frac{1}{2}E_v \\ -\frac{1}{2}G_u & E & 0 \\ \frac{1}{2}G_v & 0 & G \end{vmatrix}$$

$$=(-\tfrac{1}{2}E_v)(-\tfrac{1}{2}EG_v)+G\cdot(-\tfrac{1}{2}EE_{vv}-\tfrac{1}{2}EG_{uu}+\tfrac{1}{4}E_uG_u)$$

$$B=\begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & 0 \\ \frac{1}{2}G_u & 0 & G \end{vmatrix}$$

$$=\tfrac{1}{2}G_u(-\tfrac{1}{2}EG_u)+G(-\tfrac{1}{4}E_v^2)$$

$\therefore A-B$

$$=-\frac{(EG)^{\frac{3}{2}}}{2}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

$$\therefore K=\frac{A-B}{[EG-0]^2}=-\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

(b) when $E=G$, $F=0$,

$$K=-\frac{1}{2E}\left[\left(\frac{E_v}{E}\right)_v+\left(\frac{E_u}{E}\right)_u\right]$$

$$=-\frac{1}{2E}\left[\frac{E_{vv}E-E_v^2}{E^2}+\frac{E_{uu}E-E_u^2}{E^2}\right]$$

$$=-\frac{1}{2E^2}[E_{uu}+E_{vv}]+\frac{1}{2E^3}(E_u^2+E_v^2)$$

$$=-e^{-2f}\Delta f$$

where $f=\frac{1}{2}\lg E$

4. (a) $X(u,v) = (u\cos v,\ u\sin v,\ \log u)$

$$X_u = (\cos v,\ \sin v,\ \frac{1}{u})$$

$$X_v = (-u\sin v,\ u\cos v,\ 0)$$

$$g = \begin{bmatrix} 1+\frac{1}{u^2} & 0 \\ 0 & u^2 \end{bmatrix}$$

$$\therefore K = -\frac{1}{2\sqrt{u^2+1}}\left[\left(\frac{0}{\sqrt{u^2+1}}\right)_v + \left(\frac{2u}{\sqrt{u^2+1}}\right)_u\right]$$

$$= -\frac{1}{[u^2+1]^2}$$

(b) $Y(u,v) = (u\cos v,\ u\sin v,\ v)$

$$Y_u = (\cos v,\ \sin v,\ 0)$$

$$Y_v = (-u\sin v,\ u\cos v,\ 1)$$

$$g = \begin{bmatrix} 1 & 0 \\ 0 & u^2+1 \end{bmatrix}$$

$$\therefore K = -\frac{1}{\sqrt{u^2+1}}\left[\left(\frac{0}{\sqrt{u^2+1}}\right)_v + \left(\frac{2u}{\sqrt{u^2+1}}\right)_u\right]$$

$$= -\frac{1}{[u^2+1]^2}$$

5. $f = \frac{1}{2}\log E = \frac{1}{2}\log\left[\frac{4}{1-(x^2+y^2)}\right] = -\frac{1}{2}\log\frac{1-(x^2+y^2)}{4}$

$$\therefore f_x = -\frac{1}{2}\cdot\frac{4}{1-(x^2+y^2)}\cdot\frac{-2x}{4} = \frac{x}{1-(x^2+y^2)}$$

$$f_{xx} = \frac{1-x^2-y^2+2x^2}{[1-(x^2+y^2)]^2} = \frac{1+x^2-y^2}{[1-(x^2+y^2)]^2}$$

Similarly, $f_{yy} = \frac{1-x^2+y^2}{[1-(x^2+y^2)]^2}$

$$\therefore K = -\frac{1}{E}\Delta f = -\frac{1-(x^2+y^2)}{4}\cdot\frac{2}{[1-(x^2+y^2)]^2} = -\frac{1}{2[1-(x^2+y^2)]}$$